



Stable Kneser Graphs are almost all not weakly Hom-Idempotent

Pablo Torres, Mario Valencia-Pabon

► To cite this version:

Pablo Torres, Mario Valencia-Pabon. Stable Kneser Graphs are almost all not weakly Hom-Idempotent . 2015. hal-01119741

HAL Id: hal-01119741

<https://hal.science/hal-01119741>

Preprint submitted on 23 Feb 2015

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

Stable Kneser Graphs are almost all not weakly Hom-Idempotent*

Pablo Torres[†]

Mario Valencia-Pabon[‡]

Abstract

A graph G is said to be *hom-idempotent* if there is an homomorphism from G^2 to G , and *weakly hom-idempotent* if for some $n \geq 1$ there is a homomorphism from G^{n+1} to G^n . Larose et al. [*Eur. J. Comb.* 19:867-881, 1998] proved that Kneser graphs $KG(n, k)$ are not weakly hom-idempotent for $n \geq 2k + 1$, $k \geq 2$. We show that 2-stable Kneser graphs $KG(n, k)_{2\text{-stab}}$ are not weakly hom-idempotent, for $n \geq 2k + 2$, $k \geq 2$. Moreover, for $s, k \geq 2$, we prove that s -stable Kneser graphs $KG(ks + 1, k)_{s\text{-stab}}$ are circulant graphs and so hom-idempotent graphs.

Keywords: Cartesian product of graphs, Stable Kneser graphs, Cayley graphs, Hom-idempotent graphs.

1 Introduction

Let $[n]$ denote the set $\{1, \dots, n\}$. For positive integers $n \geq 2k$, the Kneser graph $KG(n, k)$ has as vertices the k -subsets of $[n]$ and two vertices are connected by an edge if they have empty intersection. In a famous paper, Lovász [8] showed that its chromatic number $\chi(K(n, k))$ is equal to $n - 2k + 2$. After this result, Schrijver [10] proved that the chromatic number remains the same when we consider the subgraph $KG(n, k)_{2\text{-stab}}$ of $KG(n, k)$ obtained by restricting the vertex set to the k -subsets that are 2-stable, that is, that do not contain two consecutive elements of $[n]$ (where 1 and n are considered also to be consecutive). Schrijver [10] also proved that the 2-stable Kneser graphs are *vertex critical* (or χ -critical), i.e. the chromatic number of any proper subgraph of $KG(n, k)_{2\text{-stab}}$ is strictly less than $n - 2k + 2$; for this reason, the 2-stable Kneser graphs are also known as the Schrijver graphs. After these general advances, a lot of work has been done concerning properties of Kneser graphs and stable Kneser graphs (see [2, 3, 7, 1, 9] and references therein). For example, it is well known that for $n \geq 2k + 1$ the automorphism group of the Kneser graph $KG(n, k)$ is the symmetric group induced by the permutation action on $[n]$; see [4] for a textbook account. More recently, Braun [1] showed that the automorphism group of the 2-stable Kneser graphs $KG(n, k)_{2\text{-stab}}$ is the dihedral group of order $2n$.

The *cartesian product* $G \square H$ of two graphs G and H has vertex set $V(G) \times V(H)$, two vertices being joined by an edge whenever they have one coordinate equal and the other adjacent. This product is commutative and associative up to isomorphism.

*Partially supported by MathAmSud Project 13MATH-07 (Argentina–Brazil–Chile–France) and by the French ANR JCJC CombPhysMat2Tens grant.

[†]Universidad Nacional de Rosario and CONICET, Rosario, Argentina. e-mail: ptorres@fceia.unr.edu.ar

[‡]Université Paris-13, Sorbonne Paris Cité LIPN, CNRS UMR7030, Villetaneuse, France. Currently in Délégation at the INRIA Nancy - Grand Est. e-mail: valencia@lipn.univ-paris13.fr

An *homomorphism* from a graph G into a graph H , denoted by $G \rightarrow H$, is an edge-preserving map from $V(G)$ to $V(H)$. If H is a subgraph of G and $\phi : G \rightarrow H$ has the property that $\phi(u) = u$ for every vertex u of H , then ϕ is called a *retraction* and H is called a *retract* of G . If $\phi : G \rightarrow H$ is a bijection and ϕ^{-1} is also a homomorphism from H to G , then ϕ is an *isomorphism* and we write $G \simeq H$. In particular, if $G = H$ then ϕ is an *automorphism* if and only if it is bijective. Two graphs G and H are *homomorphically equivalent*, denoted by $G \leftrightarrow H$, if $G \rightarrow H$ and $H \rightarrow G$. A graph G is called a *core* if it has no proper retracts, i.e., any homomorphism $\phi : G \rightarrow G$ is an automorphism of G . It is well known that any finite graph G is homomorphically equivalent to at least one core G^\bullet , as can be seen by selecting G^\bullet as a retract of G with a minimum number of vertices. In this way, G^\bullet is uniquely determined up to isomorphism, and it makes sense to think of it as *the core* of G . It is widely known that Kneser graphs are core. Moreover, it is not difficult to deduce that any χ -critical graph is a core. Therefore, any 2-stable Kneser graph is also a core, because they are χ -critical graphs [10].

Let A be a group and S a subset of A that is closed under inverses and does not contain the identity. The *Cayley graph* $\text{Cay}(A, S)$ is the graph whose vertex set is A , two vertices u, v being joined by an edge if $u^{-1}v \in S$. Cayley graphs of cyclic groups are often called *circulants*.

A graph G is said *vertex-transitive* if its automorphism group $\text{AUT}(G)$ acts transitively on its vertex-set. It's well known that Cayley graphs and Kneser graphs are vertex-transitive. However, 2-stable Kneser graphs are not vertex-transitive in general. For example, no automorphism in $KG(6, 2)_{2\text{-stab}}$ sends $\{1, 3\}$ to $\{1, 4\}$, since $\text{AUT}(KG(6, 2)_{2\text{-stab}})$ is the dihedral group of order 12 acting on the set $\{1, 2, \dots, 6\}$.

We write G^n for the n -fold cartesian product of a graph G . A graph G is said *hom-idempotent* if there is a homomorphism from G^2 to G , and *weakly hom-idempotent* if for some $n \geq 1$ there is a homomorphism from G^{n+1} to G^n . Larose et al. [7] showed that the Kneser graphs are not weakly hom-idempotent. However, the technique used by Larose et al. [7] cannot be extended directly to the 2-stable Kneser graphs.

A subset $S \subseteq [n]$ is *s-stable* if any two of its elements are at least "at distance s apart" on the n -cycle, that is, if $s \leq |i - j| \leq n - s$ for distinct $i, j \in S$. For $s, k \geq 2$ and $n \geq ks$, the *s-stable* Kneser graph $KG(n, k)_{s\text{-stab}}$ is the subgraph of $KG(n, k)$ obtained by restricting the vertex set of $KG(n, k)$ to the *s-stable* k -subsets of $[n]$.

In this paper, we show that almost all 2-stable Kneser graphs are not weakly hom-idempotent. Moreover, for $s, k \geq 2$, we show that *s-stable* Kneser graphs $KG(ks + 1, k)_{s\text{-stab}}$ are circulant graphs and so hom-idempotent graphs. In the sequel, we will use the term *modulo* $[n]$ to denote arithmetic operations on the set $[n]$ where n represents the 0.

2 2-stable Kneser graphs

As we have mentioned in the previous section, Braun [1] showed that the automorphism group of the 2-stable Kneser graph $KG(n, k)_{2\text{-stab}}$ is the dihedral group D_{2n} of order $2n$. We denote the elements of D_{2n} as follows (arithmetic operations are taken modulo $[n]$):

- *Rotations*: Let σ^0 be the identity permutation on $[n]$ and, for $1 \leq i \leq n - 1$, let $\sigma^i = \sigma^{i-1} \circ \sigma^1$, where σ^1 is the circular permutation $(1, 2, \dots, n - 1, n)$.
- *Reflexions*:

- Case n odd. For $1 \leq i \leq n$, let ρ_i be the permutation formed by the product of the transpositions $(i+1, i-1)(i+2, i-2) \dots (i + \frac{n-1}{2}, i - \frac{n-1}{2})$, where i is a fix point.
- Case n even. For $1 \leq i \leq \frac{n}{2}$, we have two types of reflexions: let ρ_i be the permutation formed by the product of the transpositions $(i+1, i-1)(i+2, i-2) \dots (i + \frac{n}{2} - 1, i - \frac{n}{2} + 1)$, where i and $i + \frac{n}{2}$ are fix points; and let δ_i be the permutation formed by the product of transpositions $(i, i-1)(i+1, i-2) \dots (i + \frac{n}{2} - 1, i - \frac{n}{2})$ without fix point.

An automorphism ϕ of a graph G is called a *shift* of G if $\{u, \phi(u)\} \in E(G)$ for each $u \in V(G)$. In other words, a shift of G maps every vertex to one of its neighbors [7].

Lemma 1. *Let $n \geq 2k + 2$. Then, the only two shifts of the 2-stable Kneser graph $KG(n, k)_{2\text{-stab}}$ are the rotations σ^1 and σ^{n-1} .*

Proof. It is very easy to deduce that the circular permutations $\sigma^1 = (1, 2, 3, \dots, n-1, n)$ and $\sigma^{n-1} = (\sigma^1)^{-1} = (n, n-1, n-2, \dots, 2, 1)$ are both shifts of the graph $KG(n, k)_{2\text{-stab}}$. In order to prove that they are the only two shifts of $KG(n, k)_{2\text{-stab}}$, we will proceed by cases. The arithmetic operations are taken modulo $[n]$.

- *Rotations.* Clearly, the identity permutation σ^0 is not a shift. Now, we claim that for each $i \in \{2, 3, \dots, n-2\}$, there exists a vertex v_i in $KG(n, k)_{2\text{-stab}}$ such that $\{1, i+1\} \subseteq v_i$. In fact, vertex v_i can be computed as follows:

- If $2 \leq i \leq 2k-1$ then, set $v_i = \{1, 1+2.1, 1+2.2, \dots, 1+2.(k-1)\}$ if i is even, otherwise set $v_i = \{1, 2+2.1, 2+2.2, \dots, 2+2.(k-1)\}$.
- If $2k \leq i \leq n-2$ then, set $v_i = \{1, 1+2.1, 1+2.2, \dots, 1+2.(k-2), i+1\}$.

Now, for each $2 \leq i \leq n-2$, we know that $\sigma^i(1) = 1+i$ and therefore, $1+i \in \sigma^i(v_i)$ which implies that $\{v_i, \sigma^i(v_i)\}$ is not an edge of $KG(n, k)_{2\text{-stab}}$. Thus, for $2 \leq i \leq n-2$, σ^i is not a shift of $KG(n, k)_{2\text{-stab}}$.

- *Reflexions.* We consider two cases:

- Case n odd. For each $1 \leq i \leq n$, let v_i be a vertex in $KG(n, k)_{2\text{-stab}}$ such that $i \in v_i$. Trivially, such vertex v_i always exists. Now, we know that i is a fix point under the permutation ρ_i and thus, $i \in \rho_i(v_i)$ which implies that $\{v_i, \rho_i(v_i)\}$ is not an edge of $KG(n, k)_{2\text{-stab}}$. Thus, for $1 \leq i \leq n$, ρ_i is not a shift of $KG(n, k)_{2\text{-stab}}$.
- Case n even. Analogous to the previous case, we can show that ρ_i is not a shift of $KG(n, k)_{2\text{-stab}}$, for $1 \leq i \leq \frac{n}{2}$. Now, for each $1 \leq i \leq \frac{n}{2}$, let $v_i = \{i+1, i+1+2.1, i+1+2.2, \dots, i+1+2.(k-2), i-2\}$. Clearly, v_i is a 2-stable set, since $i+1+2.(k-2)$ and $i-2$ are not consecutive integers modulo $[n]$. So, v_i is a vertex of $KG(n, k)_{2\text{-stab}}$ such that $\{i+1, i-2\} \subseteq v_i$. However, $\{i+1, i-2\} \subseteq \delta_i(v_i)$ which implies that $\{v_i, \delta_i(v_i)\}$ is not an edge of $KG(n, k)_{2\text{-stab}}$. Thus, for $1 \leq i \leq \frac{n}{2}$, δ_i is not a shift of $KG(n, k)_{2\text{-stab}}$.

□

Larose et al. [7] showed the following useful results:

Proposition 1 (Proposition 2.3 in [7]). *A graph G is hom-idempotent if and only if $G \leftrightarrow \text{Cay}(\text{Aut}(G^\bullet), S_{G^\bullet})$, where $\text{Aut}(G^\bullet)$ denotes the automorphism group of the core G^\bullet of G and S_{G^\bullet} denotes the set of all shifts of G^\bullet .*

Theorem 1 (Theorem 5.1 in [7]). *Let G be a χ -critical graph. Then G is weakly hom-idempotent if and only if it is hom-idempotent.*

Proposition 2. *Let $n \geq 2k + 2$ and let G denotes the graph $KG(n, k)_{2\text{-stab}}$. Then, $G \not\rightarrow \text{Cay}(\text{Aut}(G), S_G)$, where S_G are the shifts of G .*

Proof. We know that the automorphism group of the graph $KG(n, k)_{2\text{-stab}}$ is the dihedral group D_{2n} on $[n]$. Moreover, by Lemma 1, we known that the only two shifts of $KG(n, k)_{2\text{-stab}}$ are the circular permutations $\sigma^1 = (1, 2, \dots, n-1, n)$ and its inverse permutation $\sigma^{n-1} = (\sigma^1)^{-1} = (n, n-1, \dots, 2, 1)$. Therefore, the Cayley graph $\text{Cay}(D_{2n}, \{\sigma^1, (\sigma^1)^{-1}\})$ is a 2-regular graph, that is, each vertex (i.e. each automorphism in D_{2n}) has exactly two neighbors. In fact, notice that the vertex identity σ^0 has as neighbors the vertices σ^1 and σ^{n-1} , and as it's well known, any Cayley graph is vertex-transitive, and thus, any vertex in $\text{Cay}(D_{2n}, \{\sigma^1, (\sigma^1)^{-1}\})$ has exactly two neighbors. So, $\text{Cay}(D_{2n}, \{\sigma^1, (\sigma^1)^{-1}\})$ is a disjoint union of cycles, which implies that $2 \leq \chi(\text{Cay}(D_{2n}, \{\sigma^1, (\sigma^1)^{-1}\})) \leq 3$. Moreover, we known that $\chi(KG(n, k)_{2\text{-stab}}) = n - 2k + 2 \geq 4$, because $n \geq 2k + 2$. Therefore $KG(n, k)_{2\text{-stab}} \not\rightarrow \text{Cay}(D_{2n}, \{\sigma^1, (\sigma^1)^{-1}\})$. \square

As mentioned in the previous section, we know that any 2-stable Kneser graph is a core. Therefore, by Propositions 1 and 2, and by Theorem 1, we have the following result.

Theorem 2. *For any $n \geq 2k + 2$, the 2-stable Kneser graphs $KG(n, k)_{2\text{-stab}}$ are not weakly hom-idempotent.*

3 s -stable Kneser graphs $KG(ks + 1, k)_{s\text{-stab}}$

Let \overline{G} denote the complement graph of the graph G , i.e. \overline{G} has the same vertex set of G and two vertices are adjacent in \overline{G} if and only if they are not adjacent in G . Let p be a positive integer. The p th power of a graph G , that we denoted by $G^{(p)}$, is the graph having the same vertex set as G and where two vertices are adjacent in $G^{(p)}$ if the distance between them in G is at most equal to p , where the distance of two vertices in a graph G is the number of edges on the shortest path connecting them.

Let $n \geq 2k$ be positive integers. The Cayley graphs $\text{Cay}(\mathbb{Z}_n, \{k, k+1, \dots, n-k\})$, that we denoted by $G(n, k)$, are known as *circular graphs* [11, 6], where \mathbb{Z}_n denote the cyclic group of order n . It is well known that the Kneser graph $KG(n, k)$ contains an induced subgraph isomorphic to $G(n, k)$. In fact, let $C(n, k)$ be the subgraph of $KG(n, k)$ obtained by restricting the vertex set of $KG(n, k)$ to the shifts modulo $[n]$ of the k -subset $\{1, 2, \dots, k\}$, that is, $\{1, 2, \dots, k\}, \{2, 3, \dots, k+1\}, \dots, \{n, 1, 2, \dots, k-1\}$. Define $\phi : G(n, k) \rightarrow C(n, k)$ by putting $\phi(u) = \{u+1, u+2, \dots, u+k\}$ where the arithmetic operations are taken modulo $[n]$. Clearly, ϕ is a graph isomorphism. Notice also that the graph $G(n, k)$ is isomorphic to the graph $\overline{C_n^{(k-1)}}$, i.e. the complement graph of the $(k-1)$ th power of a cycle C_n . Vince [11] has shown that $\chi(G(n, k)) = \lceil \frac{n}{k} \rceil$.

In the remaining of this section, we will always assume w.l.o.g. that any vertex $v = \{v_1, v_2, \dots, v_k\}$ of the s -stable Kneser graph $KG(ks + 1, k)_{s\text{-stab}}$ is such that $v_1 < v_2 < \dots < v_k$, where $s, k \geq 2$. For $i \in [k-1]$, let $l_i(v) = v_{i+1} - v_i$ and $l_k(v) = v_1 + (ks + 1) - v_k$. If C is the cycle on $ks + 1$ points labeled by integers $1, 2, \dots, ks + 1$ in the clockwise direction and $v = \{v_1, v_2, \dots, v_k\}$ is a vertex of the s -stable Kneser graph $KG(ks + 1, k)_{s\text{-stab}}$, then $l_i(v)$ gives the distance in the clockwise direction between v_i and v_{i+1} in C .

Lemma 2. Let $s, k \geq 2$ and let $v = \{v_1, v_2, \dots, v_k\}$ be a vertex of $KG(k s + 1, k)_{s\text{-stab}}$. Then, $l_i(v) \in \{s, s + 1\}$ for all $i \in [k]$. Moreover, there exists exactly one $i' \in [k]$ such that $l_{i'}(v) = s + 1$.

Proof. By definition, $l_i(v) \geq s$ for any $i \in [k]$. The result follows from the fact that $\sum_{i=1}^k l_i(v) = k s + 1$. \square

Lemma 3. Let $s, k \geq 2$. The number of vertices of the graph $KG(k s + 1, k)_{s\text{-stab}}$ is equal to $k s + 1$.

Proof. Again, let C be the cycle on $k s + 1$ points labeled by integers $1, 2, \dots, k s + 1$ in the clockwise direction. From Lemma 2, we have that each vertex of $KG(k s + 1, k)_{s\text{-stab}}$ is uniquely determined by a clockwise circular interval of length $s + 1$ in C . Trivially there exist $k s + 1$ distinct clockwise circular intervals of length $s + 1$ in C and the lemma holds. \square

Proposition 3. Let $s, k \geq 2$. Then, $G(k s + 1, k) \simeq KG(k s + 1, k)_{s\text{-stab}}$.

Proof. Let C be a cycle on $k s + 1$ points. We assume that the vertices of $G(k s + 1, k)$ are disposed over C in clockwise increasing order from 0 to $k s$. By Lemmas 2 and 3, we are able to define the application $\phi : G(k s + 1, k) \rightarrow KG(k s + 1, k)_{s\text{-stab}}$ as follows: let u be a vertex of $G(k s + 1, k)$ such that $u = j k + i$, where $0 \leq j \leq s - 1$ and $0 \leq i \leq k - 1$. Then, $\phi(u) = \{u_1, \dots, u_k\}$ where,

$$u_r = \begin{cases} j + 1 + (r - 1)s, & \text{if } 1 \leq r \leq k - i \\ j + 2 + (r - 1)s, & \text{if } k - i + 1 \leq r \leq k. \end{cases}$$

Finally, define $\phi(k s) = \{s + 1, 2s + 1, \dots, k s + 1\}$. It is not difficult to prove that ϕ is a bijective function. It remains to show that ϕ is indeed a graph isomorphism. Let u, v be two vertices in $C(k s + 1, k)$. In the sequel, we assume that $v > u$. In fact, if $u > v$ we can always swap u and v . Let $u = j k + i$, where $0 \leq j \leq s - 1$ and $0 \leq i \leq k - 1$. Let $t = v - u$, where $1 \leq t \leq k - 1$. By construction, $\phi(v) \setminus \{v_p : k - i - t + 1 \leq p \leq k - i\} \subset \phi(u)$, where the arithmetic operations are taken modulo $[k]$. As $t \leq k - 1$ then, we must have that $\phi(u) \cap \phi(v) \neq \emptyset$. Analogously, let $k(s - 1) + 2 \leq v - u \leq k s$ and let $t = k s + 1 - (v - u)$. Then, $\phi(u) \setminus \{u_p : k - i - t + 1 \leq p \leq k - i\} \subset \phi(v)$, where the arithmetic operations are taken modulo $[k]$. Again, as $t \leq k - 1$ then, we must have that $\phi(u) \cap \phi(v) \neq \emptyset$. Notice that, by the previous argument, if $v - u = k$ or $v - u = k(s - 1) + 1$ then, $\phi(u) \cap \phi(v) = \emptyset$. Therefore, let $t = v - u$, where $k + 1 \leq t \leq k(s - 1)$ and assume that there exists $r \in [k]$ such that $u_r \in \phi(u) \cap \phi(v)$. We consider the following cases:

- Case $1 \leq r \leq k - i$. As $u = j k + i$ then, $v = j k + i + x k + y$, where $0 \leq j \leq s - 1$, $0 \leq i \leq k - 1$, $1 \leq x \leq s - 1$, and $0 \leq i + y \leq k - 1$. If $u_r = v_{r'}$, with $1 \leq r' \leq k - (i + y)$ then, $u_r = j + 1 + (r - 1)s = j + x + 1 + (r' - 1)s = v_{r'}$ and thus $x = (r - r')s$. As $x \geq 1$ then, $r - r' > 0$ and thus $x \geq s$ which is a contradiction. If $k - (i + y) + 1 \leq r' \leq k$ then, $j + 1 + (r - 1)s = j + x + 2 + (r' - 1)s$ which implies that $x = (r - r')s - 1$. However, $r \leq k - i$ and $r' \geq k - i - y + 1$ and so, $x \leq (y - 1)s - 1$. Moreover, as $x \geq 1$ then, $y \geq 2$ and thus $x = s - 1$, but it implies that $t \geq k(s - 1) + 2$ which is a contradiction.
- Case $k - i + 1 \leq r \leq k$. As the previous case, $v = (j + x)k + (i + y)$, with $1 \leq x \leq s - 1$ and $0 \leq i + y \leq k - 1$. If $u_r = j + 2 + (r - 1)s = j + x + 1 + (r' - 1)s = v_{r'}$, with $1 \leq r' \leq k - (i + y)$ then, $x = (r - r')s + 1$ which implies that $x \geq (y + 1)s + 1 \geq s$, that is a contradiction. If $k - (i + y) + 1 \leq r' \leq k$ then, $j + 2 + (r - 1)s = j + x + 2 + (r' - 1)s$. Then, $x = (r - r')s$ and as $x \geq 1$ then $r - r' > 0$, but it implies that $x \geq s$ which is again a contradiction.

Therefore, vertices u, v in $C(k s + 1, k)$ are adjacent if and only if vertices $\phi(u), \phi(v)$ in $KG(k s + 1, k)_{s\text{-stab}}$ are adjacent. \square

Notice that if $n \geq 2r + 3$ the automorphism group of the r^{th} power of a n -cycle, $C_n^{(r)}$, is the dihedral group D_{2n} (see Remark 1) and thus, $\text{AUT}(C_n^{(r)})$ is also D_{2n} .

Remark 1. For $r \geq 1$ and $n \geq 2r + 3$ the automorphism group of $C_n^{(r)}$ is the dihedral group D_{2n} .

In fact, it is not hard to see that D_{2n} is a subgroup of $\text{AUT}(C_n^{(r)})$. Let $\alpha \in \text{AUT}(C_n^{(r)})$ and let $N[i]$ denote the closed neighborhood of a vertex i in $C_n^{(r)}$. Assume that α sends two consecutive vertices i, j in the cycle to non consecutive vertices i', j' in the cycle, respectively. Observe that $|N[i] \cap N[j]| = 2r$, but $|N[i'] \cap N[j']| \leq 2r - 1$ since there exists at least two vertices in $N[i'] \setminus N[j']$. Therefore, any automorphism of $C_n^{(r)}$ sends consecutive vertices to consecutive vertices. From this fact it is easy to see that any automorphism of $C_n^{(r)}$ belongs to D_{2n} .

Therefore, by Proposition 3, the automorphism group of the graph $KG(k s + 1, k)_{s\text{-stab}}$ is the dihedral group $D_{2(k s + 1)}$. However, the problem to compute the automorphism group of the graph $KG(n, k)_{s\text{-stab}}$ for $k \geq 2$, $s > 2$, and $n > k s + 1$ is still open.

Another direct consequence of Proposition 3 is that $\chi(KG(k s + 1, k)_{s\text{-stab}}) = s + 1$. This fact has been found also by Meunier (see Proposition 1 in [9]).

Let $\text{Cay}(A, S)$ be a Cayley graph. If $a^{-1}Sa = S$ for all $a \in A$, then $\text{Cay}(A, S)$ is called a *normal Cayley graph*.

Lemma 4 ([5]). *Any normal Cayley graph is hom-idempotent.*

Note that all Cayley graphs on abelian groups are normal, and thus hom-idempotents. In particular, the *circulant* graphs are Cayley graphs on cyclic groups (i.e., cycles, powers of cycles, complements of powers of cycles, complete graphs, etc). Therefore, by Proposition 3 and Lemma 4, we have the following result.

Theorem 3. *Let $s, k \geq 2$. Then, $KG(k s + 1, k)_{s\text{-stab}}$ is hom-idempotent.*

References

- [1] B. Braun. Symmetries of the stable Kneser graphs. *Advances in Applied Mathematics*, 45:12–14, 2010.
- [2] P. Frankl. On the chromatic number of the general Kneser graph. *Journal of Graph Theory*, 9(2):217–220, 1985.
- [3] P. Frankl, Z. Füredi. Extremal problems concerning Kneser graphs. *Journal of Combinatorial Theory Ser. B*, 40(3):270–284, 1986.
- [4] C. D. Godsil, G. Royle. *Algebraic graph theory*. Graduate Texts in Mathematics. Springer, 2001.

- [5] G. Hahn, P. Hell, S. Poljak. On the ultimate independence ratio of a graph. *European Journal on Combinatorics*, 16:253–261, 1995.
- [6] G. Hahn, C. Tardif. Graph homomorphisms: structure and symmetry. In *Graph Symmetry, Algebraic Methods and Applications*, NATO ASI Ser. C 497:107-166, 1997.
- [7] B. Larose, F. Laviolette, C. Tardif. On normal Cayley graphs and Hom-idempotent graphs, *European Journal of Combinatorics*, 19:867–881, 1998.
- [8] L. Lovász. Kneser’s conjecture, chromatic number and homotopy, *Journal of Combinatorial Theory, Series A*, 25:319-324, 1978.
- [9] F. Meunier. The chromatic number of almost stable Kneser hypergraphs. *Journal of Combinatorial Theory, Series A*, 118:1820–1828, 2011.
- [10] A. Schrijver. Vertex-critical subgraphs of Kneser graphs. *Nieuw Arch. Wiskd.*, 26(3):454–461, 1978.
- [11] A. Vince. Star chromatic number. *Journal of Graph Theory*, 12(4):551-559, 1988.